JOURNAL OF DIFFERENTIAL EQUATIONS 10, 181–187 (1971)

A Nonlinear Sturm-Liouville Problem

MANUEL F. ESTABROOKS AND JACK W. MACKI

Department of Mathematics, The University of Alberta
Edmonton, Alberta, Canada

Received June 13, 1969

Our purpose is to extend the classical polar-coordinate technique (Prufer technique) to nonlinear problems of a special type. These investigations were stimulated by a paper of Bihari [1]. Results of a similar nature for a different type of equation can be found in the papers of P. Bailey [4], F. Brauer [5], and R. Moroney [6]. Brauer gives an extensive bibliography.

Consider the problem

$$(P_\lambda)\qquad \begin{aligned} &(p(x)y')' + F(\lambda, x, y, y') = 0,\\ &\alpha y(a) + \beta y'(a) = 0, \qquad \gamma y(b) + \delta y'(b) = 0, \end{aligned}$$

where $\alpha^2 + \beta^2 > 0$, $\gamma^2 + \delta^2 > 0$, $[a, b]$ is a compact interval, and (if R stands for the reals):

(1) *$p(x)$ is integrable on $[a, b]$, and $0 < c^{-1} \leqslant p(x) \leqslant d < +\infty$.*

(2) *$F(\lambda, x, 0, 0) = 0, F(\lambda, x, u, v) \in C(D_0)$, where*

$$D_0 = R_0 \times [a, b] \times R \times R, \qquad R_0 = \{\lambda \mid \lambda \geqslant 0\},$$

and only the trivial solution of the equation in P_λ satisfies $y(x) = y'(x) = 0$ for any x in $[a, b]$.

(3) *$uF(\lambda, x, u, v) > 0$ for all $u \neq 0$, $\lambda > 0$, $(\lambda, x, u, v) \in D_0$.*

(4) *For each $\Delta > 0$ and each $x \in [a, b]$, set*

$$S(\Delta) = \{(u, v) \mid u^2 + p^2(x)\, v^2 = 1, \mid u \mid \geqslant \Delta\}.$$

Then $\inf_{S(\Delta)} uF(\lambda, x, u, v) = G_\Delta(\lambda, x)$ is integrable on $[a, b]$ for each $\lambda > 0$, and for Δ sufficiently small

$$\limsup_{\lambda \to +\infty} G_\Delta(\lambda, x) = +\infty, \qquad \textit{uniformly on } [a, b],$$

505/10/2-1

(5) *For any* $(\lambda, x, u, v) \in D_0$ *for which* $\rho = (u^2 + p^2(x)\, v^2)^{1/2} > 0$, *we have*

$$F(\lambda, x, u, v) = \phi(\rho)\, F(\lambda, x, u/\rho, v/\rho), \qquad \rho = (u^2 + p^2(x)\, v^2)^{1/2},$$

where

$$(*) \quad \phi \in C(0, \infty), \frac{\phi(t)}{t} \geqslant \eta > 0 \qquad \textit{for} \quad t \in (0, \infty), \int_1^{\infty} \frac{du}{\phi(u)} = +\infty.$$

Assumption (5) means, intuitively, that F can be defined subject to (1)–(4) on the ellipse $u^2 + p^2(x)\, v^2 = 1$ as a function of (λ, x), and then defined throughout the (u, v) plane by the projection formula above, using any ϕ satisfying the conditions (*). An example of such a function is

$$F(\lambda, x, u, v) = \rho \log(e - 1 + \rho)\, e^x(1 + \lambda + \lambda \cos \lambda) \sin \frac{u}{\rho},$$
$$\phi(\rho) = \rho \log(e - 1 + \rho),$$

where $\rho = (u^2 + p^2(x)\, v^2)^{1/2}$. The first two conditions on ϕ, as listed in (*), are stated in the form in which we shall apply them. The third condition, when combined with the other two, guarantees that each positive solution of

$$y' = cy + K\phi(y)$$

is extendable to $[a, b]$ for any positive constants c, K. (For a proof, see Hartman [3, p. 30]). We have not stated this extendability property as an assumption, because the third condition in (*) (given the other two) is necessary as well as sufficient for this extendability. We shall not need this fact in the analysis to follow, so we omit the simple proof.

Note that the above assumptions are satisfied in the classical case [2, p. 398]. Also, Bihari's assumptions in [1] imply the above assumptions. The essential difference from Bihari's assumptions is in Assumption (5)—he assumes F is positive homogeneous of degree one in (u, v)—although we have moderately weakened some other assumptions as well. We have so far made assumptions only for $\lambda \geqslant 0$; we consider $\lambda \in R$ in Corollary 1.

THEOREM. *Under Assumptions* (1)–(5) *above, we can select an infinite sequence of eigenvalues of* P_λ, $0 \leqslant \lambda_m < \lambda_{m+1} < \cdots$, *indexed so that any eigenfunction for* λ_k *has exactly k zeros on* (a, b), *and such that* $\lim_{k\to\infty} \lambda_k = +\infty$. *Furthermore, if* λ *is any positive eigenvalue of* P_λ, *then* $\lambda \in (\lambda_k, \lambda_{k+1})$ *for some* $k \geqslant m$, *and each eigenfunction for* λ *has no more than* $k + 1$ *zeros on* (a, b).

We shall prove the theorem by establishing a sequence of lemmas. It should be mentioned that our theorem provides no information as to whether the spectrum or any of its subsets is continuous or discrete, nor does it say anything about the multiplicity of eigenfunctions.

LEMMA 1. *Assume that* (1), (2), *and* (5) *hold, and let* $\lambda \in R_0$. *Then any solution of the initial value problem obtained by deleting the condition at b from* P_λ *is extendable to* $[a, b]$.

Proof. We have, if $y(x)$ is a solution on (a, ω_+),

$$|(py')'| = |F(\lambda, x, y, y')| \leqslant \phi(\rho) \, |F(\lambda, x, \xi, \eta)|,$$

where $\rho = (y^2 + (py')^2)^{1/2}$, $\xi = y/\rho$, $\eta = y'/\rho$. Thus

$$(py')' \leqslant K(\lambda) \, \phi(\rho), \qquad K(\lambda) = \sup |F(\lambda, x, \xi, \eta)|,$$

where the sup is taken over $\{\lambda\} \times [a, b] \times \{(\xi, \eta) \mid \xi^2 + \eta^2 p^2(x) = 1, x \in [a, b]\}$, a compact subset of D_0. Then ρ is differentiable on (a, ω_+) by Assumption (2), and

$$|\rho\rho'| = |yy' + (py')(py')'| \leqslant \rho(|y'|^2 + [(py')']^2)^{1/2} \leqslant \rho[c^2\rho^2 + K^2\phi^2(\rho)]^{1/2},$$

so $|\rho'| \leqslant c\rho + K\phi(\rho)$, and ρ is bounded above by a maximal solution of $\rho' = c\rho + K\phi(\rho)$ [3, p. 26]; so ρ is extendable to $[a, b]$.

Remark. If D_0 is replaced by $D = R \times [a, b] \times R \times R$ in (2) and (5), then the conclusion of the above lemma is valid for any $\lambda \in R$.

We now use the polar coordinate technique to convert P_λ to an equivalent problem. For any solution $y(x, \lambda)$ of the equation in P_λ, set $y(x, \lambda) = \rho(x, \lambda) \sin \theta(x, \lambda)$, $p(x) \, y'(x, \lambda) = \rho(x, \lambda) \cos \theta(x, \lambda)$, that is, $\rho^2 = y^2 + (py')^2$, $\theta = \arctan(y/py')$, where we may use any convenient branch of the arctan. The following problem is then equivalent, using the given transformations, to the problem P_λ:

$$\rho' = \left[(1/p(x)) \cos \theta \sin \theta - F\left(\lambda, x, \sin \theta, \frac{\cos \theta}{p(x)}\right) \frac{\phi(\rho)}{\rho} \cos \theta\right] \rho$$

$$(Q_\lambda) \qquad \theta' = (1/p(x)) \cos^2 \theta + F\left(\lambda, x, \sin \theta, \frac{\cos \theta}{p(x)}\right) \frac{\phi(\rho)}{\rho} \sin \theta,$$

$$\theta(a, \lambda) = \operatorname{Arctan}(-\beta/\alpha p(a)), \ \theta(b, \lambda) = \operatorname{Arctan}(-\delta/\gamma p(b)) + n\pi,$$

where n is any integer, and we use Arctan θ for that branch of arctan (discontinuous at 0) for which $0 \leqslant \operatorname{Arctan} \theta < \pi$. Note that $\theta' \geqslant 0$ for $\lambda \geqslant 0$. If α or β is zero, we define the respective Arctan as $\pi/2$.

LEMMA 2. *Suppose* (1)–(5) *hold, and let* (ρ, θ) *be any solution of the initial value problem obtained by deleting the condition at* $x = b$ *in* Q_λ. *Then*

$$\limsup_{\lambda \to +\infty} \theta(b, \lambda) = +\infty.$$

Proof. We must first show that (ρ, θ) is extendable to $[a, b]$. $\rho(x)$ is clearly no problem, by its definition and Lemma 1. Since $\phi(\rho)/\rho$ is then bounded, and

$$0 \leqslant \theta' \leqslant c + K\phi(\rho)/\rho,$$

we see that θ is indeed extendable.

Now let $\lambda > 0$. If δ_1 is sufficiently small, then the following implications are valid for any nonnegative integer n:

$$|\theta(x, \lambda) - n\pi| \leqslant \delta_1 \Rightarrow \theta'(x, \lambda) \geqslant c \cos^2 \delta_1 ;$$
$$|\theta(x, \lambda) - n\pi| \geqslant \delta_1 \Rightarrow \theta'(x, \lambda) \geqslant \eta G_\Delta(\lambda, x),$$

where $\Delta = \sin \delta_1$, $\eta = \inf_{\rho > 0} (\phi(\rho)/\rho)$. Let N be a given natural number. The first estimate above implies that if $\delta_1 > 0$ is given, we can choose $\epsilon > 0$ such that

$$|\theta(x, \lambda) - n\pi| \leqslant \delta_1 \Rightarrow \theta(x + \epsilon, \lambda) \geqslant n\pi + \delta_1, \qquad x \in [a, b],$$

e.g., we can choose $\epsilon = 2\delta_1/c \cos^2 \delta_1$. If $\theta(a, \lambda) > 0$, we choose $\delta_1 < \theta(a, \lambda)$ to conclude that $\theta(x, \lambda) > \delta_1$ on $[a, b]$, in particular at $x_1 = a + (b - a)/N$. If $\theta(a, \lambda) = 0$, we note that $\theta'(a, \lambda) > 0$; so again we can find a δ_1 such that $\theta(x_1, \lambda) > \delta_1$. We can then select λ_N so large that

$$\theta(x_2 - \epsilon, \lambda_N) \geqslant \pi - \delta_1, \qquad x_2 = a + 2(b - a)/N,$$

so that $\theta(x_2, \lambda_N) \geqslant \pi + \delta_1$.

Using the second estimate on θ' above and our assumption on $G_\Delta(\lambda, x)$. An easy induction now shows that $\theta(b, \lambda_N) > (N - 1)\pi$, thus

$$\limsup_{\lambda \to +\infty} \theta(b, \lambda) = +\infty.$$

Proof of the Theorem. Our assumptions guarantee that $\theta(b, \lambda)$ is a continuous function of λ, and, since $\limsup_{\lambda \to +\infty} \theta(b, \lambda) = +\infty$, we can conclude that $\theta(b, \lambda)$ takes on every value greater than $\inf_{\lambda \geqslant 0} \theta(b, \lambda)$. This inf exists and is nonnegative, since $\theta'(a, \lambda) \geqslant 0$ for all $\lambda \in R$ and $\theta(x, \lambda)$ cannot cross the x-axis on $[a, b]$ (recall $\theta' > 0$ when $\theta = n\pi$). We set

$$A = \{k \mid \operatorname{Arctan}(-\delta/\gamma p(b) + k\pi \geqslant \inf_{\lambda \geqslant 0} \theta(b, \lambda)\} = \{m, m + 1, \ldots\},$$

and define

$$\lambda_m = \inf\{\lambda \mid \theta(b, \lambda) = \operatorname{Arctan}(-\delta/\gamma p(b)) + m\pi, \quad \lambda \geqslant 0\},$$

$$\lambda_k = \inf\{\lambda \mid \theta(b, \lambda) = \operatorname{Arctan}(-\delta/\gamma p(b)) + k\pi, \quad \lambda \geqslant \lambda_{k-1}\},$$

for $k > m$. Clearly $\theta(b, \lambda_k) = \operatorname{Arctan}(-\delta/\gamma p(b)) + k\pi$ for each $k \in A$, since θ is continuous in λ. Any nonnegative eigenvalue of P_λ must be greater than λ_m, by the definition of λ_m. Furthermore, $\lim_{k\to\infty} \lambda_k = +\infty$, for if not, then $\lim_{k\to\infty} \lambda_k = \lambda^*$, and

$$+\infty = \lim_{k\to\infty} [\operatorname{Arctan}(-\delta/\gamma p(b)) + k\pi] = \lim_{k\to\infty} \theta(b, \lambda_k) = \theta(b, \lambda^*),$$

in contradiction to the extendability of $\theta(x, \lambda^*)$.

Since $\theta' > 0$ whenever $\theta = n\pi$, and $y(x, \lambda) = 0$ if and only if $\theta(x, \lambda) = n\pi$, it is clear that each zero of an eigenfunction is simple, and that $y(x, \lambda_k)$ has exactly k zeros on (a, b). For example, the first eigenvalue λ_m, is such that $\operatorname{Arctan}(-\delta/\gamma p(b)) + m\pi = \theta(b, \lambda_m)$, which means that $\theta(x, \lambda_m)$ has crossed each of the lines $\theta = \pi, \theta = 2\pi, \ldots, \theta = m\pi$, exactly once, yielding exactly m zeros. Now suppose that λ is a positive eigenvalue, $\lambda \in (\lambda_k, \lambda_{k+1})$. Then $\theta(b, \lambda) < \operatorname{Arctan}(-\delta/\gamma p(b)) + (k + 1)\pi$, from the definition of λ_{k+1}, so $\theta(b, \lambda)$ can cross at *most* the lines $\theta = \pi, \theta = 2\pi, \ldots, \theta = k\pi$, to give at most k zeros to any of its eigenfunctions.

COROLLARY 1. *If, in addition to (1)–(5) above, we assume that* $uF(\lambda, x, u, v) < 0$ *for* $u \neq 0$ *and* $\lambda < 0$, *and that for each* $\lambda < 0$ *and* $\Delta > 0$, $\sup_{S(\Delta)} [uF(\lambda, x, u, v)] = H_\Delta(\lambda, x)$ $(\leqslant 0)$ *satisfies* $\liminf_{\lambda\to-\infty} H_\Delta(\lambda, x) = -\infty$ *uniformly on* $[a, b]$, *then the problem* P_λ *has an infinite sequence of eigenvalues,* $\lambda_0 < \lambda_1 < \cdots$, *with* $\lim_{k\to\infty} \lambda_k = +\infty$, *such that the eigenfunction corresponding to* λ_k *has exactly k zeros in* (a, b). *If* λ *is any eigenvalue of* P_λ *satisfying* $\lambda \geqslant \lambda_0$, *then* $\lambda \in (\lambda_k, \lambda_{k+1})$ *for some k, and any eigenfunction corresponding to* λ *has at most* $k + 1$ *zeros in* (a, b).

Remark. Again we emphasize that we can say nothing about the discreteness or continuity of the spectrum and its subsets. We do not claim that λ_0 is the least eigenvalue for P_λ (see Corollary 2).

Proof. We shall show that $\liminf_{\lambda\to-\infty} \theta(b, \lambda) = 0$, where θ is any solution of the problem Q_λ with the condition at b deleted. To do this, we show that, given any $\epsilon > 0$, we can find a λ_ϵ for which $\theta(b, \lambda_\epsilon) < \epsilon$. We may assume without loss of generality that $\epsilon < \pi - \theta(a, \lambda)$ (recall that $\theta(a, \lambda)$ is independent of λ). We give only the rough details, since the proof is very similar to the classical linear case [2, p. 398]. We set $\Delta = \sin \epsilon$, and pick

$\{\lambda_k\} \to -\infty$ such that $\eta H_\Delta(\lambda_k, x) < -c$ on $[a, b]$, $\lim_{k\to\infty} H_\Delta(\lambda_k, x) = -\infty$. Then

$$\pi - \epsilon \geqslant \theta(x, \lambda_k) \geqslant \epsilon \Rightarrow \theta'(x, \lambda_k) \leqslant c + \eta H_\Delta(\lambda_k, x) < 0.$$

Thus, since $\theta(a, \lambda_k) < \pi - \epsilon$, $\theta(x, \lambda_k) < \pi - \epsilon$ in $[a, b]$. Also, if $\theta(x_0, \lambda_k) \leqslant \epsilon$ for some x_0, then $\theta(b, \lambda_k) \leqslant \epsilon$. Now suppose $\theta(x, \lambda_k) \geqslant \epsilon$ for all $x \in [a, b]$, all k. Then we can integrate the above estimate on θ' to get

$$-\pi \leqslant \theta(b, \lambda_k) - \theta(a, \lambda_k) \leqslant c(b - a) + \eta \int_a^b H_\Delta(\lambda_k, x)\, dx.$$

Since $\lim_{k\to\infty} H_\Delta(\lambda_k, x) = -\infty$ uniformly on $[a, b]$, this is a contradiction, so we must conclude that $\liminf_{\lambda\to-\infty} \theta(b, \lambda) = 0$. Thus $\theta(b, \lambda)$ takes on all values in $(0, \infty)$, and we may proceed as in the proof of the theorem, setting (for $\delta \neq 0$) λ_0 to be any value of λ for which

$$\theta(b, \lambda_0) = \operatorname{Arctan}(-\delta/\gamma p(b)),$$

and defining, for $k = 1, 2, \ldots$,

$$\lambda_k = \inf\{\lambda \mid \theta(b, \lambda) = \operatorname{Arctan}(-\delta/\gamma p(b)) + k\pi, \quad \lambda \geqslant \lambda_{k-1}\}.$$

In the singular case $\delta = 0$ (requiring $\theta(b, \lambda_0) = 0$) we define λ_0 to be any value of λ for which $\theta(b, \lambda_0) = \pi$. We cannot take λ_0 to be

$$\inf\{\lambda \mid \theta(b, \lambda) = \operatorname{Arctan}(-\delta/\gamma p(b))\},$$

since nothing in our analysis guarantees that this inf exists.

COROLLARY 2. *If the hypotheses of Corollary 1 are strengthened by requiring*

$$\text{(a)} \quad \lim_{\lambda\to\infty} H_\Delta(\lambda, x) = -\infty, \qquad \textit{and} \qquad \text{(b)} \quad \delta \neq 0,$$

then each eigenvalue λ of P_λ belongs to some interval $(\lambda_k, \lambda_{k+1})$, and its corresponding eigenfunctions have at most $k + 1$ zeros on (a, b).

Proof. In this case, we can parallel the proof of Corollary 1 to show $\lim_{\lambda\to-\infty} \theta(b, \lambda) = 0$. Thus (since $\delta \neq 0$)

$$\lambda_0 = \inf\{\lambda \mid \theta(b, \lambda) = \operatorname{Arctan}(-\delta/\gamma p(b))\}$$

exists. Then if λ is an eigenvalue of P_λ, $\lambda \geqslant \lambda_0$ and so $\lambda \in (\lambda_k, \lambda_{k+1})$ for some k. The rest is clear from the proof of the theorem.

EXAMPLE. Let $p(x) \equiv 1$, $q(x) > 0$,

$$F(\lambda, x, y, y') = (\lambda + 1 + \lambda \cos \lambda)\, q(x) y \exp \left[- \frac{y^2 + (y')^2 - 1}{y^2 + (y')^2 + 1} \right].$$

The Assumptions (1)–(5) hold, with

$$\phi(\rho) = \rho \exp \left(- \frac{\rho^2 - 1}{\rho^2 + 1} \right).$$

Thus the set of positive eigenvalues of P_λ, with this particular F, possesses a discrete subset $\{\lambda_k\}$, with $\lim_{k\to\infty} \lambda_k = +\infty$, which can be indexed as $\lambda_m < \lambda_{m+1} < \cdots$ so any eigenfunction corresponding to λ_k has exactly k zeros. This F also satisfies the assumptions of Corollary 1, so the conclusions of the corollary apply. Corollary 2 does not apply, since

$$H_\Delta(\lambda, x) = \sup_{S(\Delta)} [uF(\lambda, x, u, v)] = (\lambda + 1 + \lambda \cos \lambda)\, q(x) \Delta^2,$$

$$-\infty = \liminf_{\lambda\to-\infty} H_\Delta(\lambda, x) < \limsup_{\lambda\to-\infty} H_\Delta(\lambda, x) = +1.$$

ACKNOWLEDGMENT

This research was supported, in part, by the Defense Research Board of Canada under Grant Number DRB-9540-11.

REFERENCES

1. I. BIHARI, Ausdehnung der Sturmschen Oscillations- und Vergleichssatzen auf die Lösungen gewisser nicht-linearer Differentialgleichungen zweiter Ordnung, *Publ. Math. Inst., Hungarian Acad. Sci.* **2** (1957), 159–173.
2. E. HILLE, "Lectures on Ordinary Differential Equations," Addison-Wesley, Reading, Mass., 1969.
3. P. HARTMAN, "Ordinary Differential Equations," Wiley, New York, 1964.
4. PAUL B. BAILEY, An eigenvalue theorem for nonlinear second-order differential equations, *J. Math. Anal. Appl.* **20** (1967), 94–102.
5. F. BRAUER, Nonlinear perturbations of Sturm–Liouville boundary value problems, *J. Math. Anal. Appl.* **22** (1968), 591–598.
6. R. M. MORONEY, A class of characteristic value problems, *Trans. Amer. Math. Soc.* **102** (1962), 446–470.